ON THE CONNECTIVITY OF THE PROPER ENHANCED POWER GRAPH OF GROUPS

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Abstract. Let G be a group. The *enhanced power graph* of G, denoted by $\mathcal{P}_e(G)$, is a simple undirected graph with the vertex set is G, in which two distinct vertices x and y are adjacent if and only if there exists an element $z \in G$ such that both x and y are powers of z. The *proper enhanced power graph* $\mathcal{P}_e^*(G)$ of G is an induced subgraph of $\mathcal{P}_e(G)$ generated by all nonidentity elements of G. In this paper, we show that $\mathcal{P}_e(G)$ is connected if and only if G is periodic. We also show that if G is a p-group, then $\mathcal{P}_e^*(G)$ is connected if and only if G has a unique subgroup of order p. For an arbitrary group G, we show that G is either periodic or torsion-free provided that $\mathcal{P}_e^*(G)$ is connected. As an application, we show that if D is a division ring of characteristic not 2 containing more than 3 elements, then $\mathcal{P}_e^*(\mathrm{GL}_n(D))$ of the general skew linear group $\mathrm{GL}_n(D)$ is connected if and only if D is a locally finite field and $\mathrm{GL}_n(D)$ is different from $\mathrm{GL}_2(\mathbb{F}_9)$.

1. Introduction

The *enhanced power graph* of a group G, denoted by $\mathcal{P}_e(G)$, is a simple undirected graph with the vertex set is G, in which two distinct vertices x and y are adjacent if and only if there exists an element $z \in G$ such that both x and y are powers of z. In other words, x and y are adjacent in $\mathcal{P}_e(G)$ if and only if the subgroup $\langle x, y\rangle$ generated by x and y is cyclic. The concept of an enhanced power graph was first introduced by Aalipour *et al.* [1]. The enhanced power graphs of groups have been studied by several authors (see [4],[5],[15],[20],[23],[26]). The *proper enhanced power graph* $\mathcal{P}_e^*(G)$ of G is an induced subgraph of $\mathcal{P}_e(G)$ generated by all nonidentity elements of G. In this paper, we study the connectivity of the proper enhanced power graph $\mathcal{P}_e^*(\mathrm{GL}_n(D))$ of the skew general linear group $\mathrm{GL}_n(D)$.

For a simple undirected graph Γ, a edge $\{x, y\}$ in Γ is denoted by $x \sim y$, and we say that x is adjacent to y. A graph Γ is called *complete* if $x \sim y$ for every distinct vertices pair x and y. A finite sequence of distinct vertices $x_0, x_1, \ldots, x_{n-1}, x_n$ in Γ is called a *path* from x_0 to x_n if $x_i \sim x_{i+1}$ for every $0 \leq i \leq n$. The number n is called the *length* of this path. Two distinct vertices x and y are *connected* if there exits a path from x to y. A graph Γ is said to be *connected* if x and y are connected for every $x, y \in \Gamma$. If x and y are connected, the *distance* between two vertices x and y is the length of the shortest path from x to y, and is denoted by $d(x, y)$, otherwise, we define $d(x, y) = \infty$. The *diameter* of a graph Γ is defined

$$\mathrm{diam}(\Gamma) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } \Gamma\}.$$

A group G is called *periodic* if every element of G has finite order. A group G is said to be *torsion-free* if every nonidentity element of G has infinite order. A group G is called *locally cyclic* if every finitely generated subgroup of G is cyclic. A group is *locally finite* if any finitely generated subgroup is finite. Let p be a prime number.

A group G is said to be *p-group* if the order of every element of G is a power of p. Observe that every p-group is periodic. The *generalized quaternion group* Q_{2^n}, $n \geq 2$, is defined by

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^4 = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1} \rangle.$$

For $n = 3$, the group Q_8 is called a *quaternion group*.

Recall that the *power graph* $\mathcal{P}(G)$ of a group G is a simple undirected graph whose the vertex set is G, and two distinct vertices x and y are adjacent if one is a positive power of the other. In other words, x and y are adjacent if $\langle x \rangle \leq \langle y \rangle$ or $\langle y \rangle \leq \langle x \rangle$. This graph was first introduced by Chakraborty *et al.* [10]. The *proper power graph* $\mathcal{P}^*(G)$ of a group G is the induced subgraph of $\mathcal{P}(G)$ generated by all nonidentity elements of G. Some results on the power graphs and the proper power graphs, we prefer to papers [1],[2],[3],[5],[6],[7],[8],[9],[11],[13],[17],[18],[21],[22]. Observe that the power graph is a spanning subgraph of the enhanced power graph, and so the proper power graph is a spanning subgraph of the power enhanced power graph. In [1], the authors characterized the finite groups for which equality holds for the power graph and enhanced power graph. They proved that if G is a finite group, then $\mathcal{P}(G)$ is equal to $\mathcal{P}_e(G)$ if and only if every element of G has prime power order. The *cyclic subgroup intersection graph* of a group G, denoted by $\Gamma_c(G)$, is a simple undirected graph whose the vertex set consists of all non-trivial proper cyclic subgroups of G, and two distinct vertices $\langle x \rangle$ and $\langle y \rangle$ are adjacent if and only if $\langle x \rangle \cap \langle y \rangle$ is nontrivial. Some results on the cyclic subgroup intersection graphs of groups, we prefer to papers [14] and [24].

In this paper, we study the connectivity of the graphs $\mathcal{P}_e(G)$ and $\mathcal{P}_e^*(G)$ of a group G. We also show the close relation between three graphs, namely, $\mathcal{P}_e^*(G)$, $\mathcal{P}^*(G)$, and $\Gamma_c(G)$. For a nontrivial p-group G, we show that $\mathcal{P}_e(G) = \mathcal{P}(G)$, and that $\mathcal{P}_e^*(G)$ is connected if and only if G has a unique subgroup of order p. In the case when G has finite order p^n, we prove that $\mathcal{P}_e^*(G)$ is connected if and only if G is either cyclic or a 2-group isomorphic to Q_{2^n}. For an arbitrary group G, we show that $\mathcal{P}_e(G)$ is connected if and only if G is periodic. We also prove that if $\mathcal{P}_e^*(G)$ is connected, then G is either periodic or torsion-free. For an infinite division ring D, we prove that $\mathcal{P}_e^*(\mathrm{GL}_n(D))$ is connected if and only if D is a locally finite field provided the characteristic of D is not 2.

2. The connectivity of enhanced power graphs of arbitrary groups

In this section, we investigate the connectivity of the enhanced power graph $\mathcal{P}_e(G)$ of a group G and compute its diameter. We also show the close relation between the enhanced power graph $\mathcal{P}_e(G)$ and the power graph $\mathcal{P}(G)$. Observe that $\mathcal{P}(G)$ is a spanning subgraph of $\mathcal{P}_e(G)$. However, $\mathcal{P}(G)$ is not equal to $\mathcal{P}_e(G)$ in general. Here, we show that if G is a p-group, then $\mathcal{P}_e(G) = \mathcal{P}(G)$, and that if F is a field, then F is locally finite if and only if $\mathcal{P}_e(F^*) = \mathcal{P}(F^*)$.

For a finite group G, it was shown in [4, Theorem 2.4] that $\mathcal{P}_e(G)$ is complete if and only if G is cyclic. In the following proposition, we get a result for arbitrary groups.

Proposition 2.1. *Let G be a group. Then, the graph $\mathcal{P}_e(G)$ is complete if and only if G is locally cyclic.*

Theorem 2.2. *Let G be a group. Then, the following assertions hold:*

(1) *The graph* $\mathcal{P}_e(G)$ *is connected and* $\operatorname{diam}(\mathcal{P}_e(G)) \leq 2$.
(2) $\operatorname{diam}(\mathcal{P}_e(G)) = 1$ *if and only if* G *is locally cyclic.*
(3) $\operatorname{diam}(\mathcal{P}_e(G)) = 2$ *if and only if* G *is noncyclic periodic.*

Proof. Assume that x and y are two distinct vertices in $\mathcal{P}_e(G)$. If $\langle x, y\rangle$ is cyclic, then $x \sim y$, and so $\operatorname{diam}(\mathcal{P}_e(G)) = 1$. Hence, we can suppose that $\langle x, y\rangle$ is not cyclic. Observe that $\langle x, 1\rangle$ is cyclic, which implies that $x \sim 1$. Similarly, $1 \sim y$, and we get a path $x \sim 1 \sim y$ in $\mathcal{P}_e(G)$. Therefore, we can conclude that $\mathcal{P}_e(G)$ is connected and $\operatorname{diam}(\mathcal{P}_e(G)) \leq 2$. Now, we have $\operatorname{diam}(\mathcal{P}_e(G)) = 1$ if and only if $\mathcal{P}_e(G)$ is complete. Equivalently, G is locally cyclic by Proposition 2.1. Finally, assume that $\operatorname{diam}(\mathcal{P}_e(G)) = 2$, and $a \in G\backslash\{1\}$. Then, there exists $b \in G$ such that $1 \sim b \sim a$ is a path in $\mathcal{P}_e(G)$. It is easy to verify a has finite order. Hence, G is periodic. Clearly, G is noncyclic. The converse is clear. □

Theorem 2.3. *Let* G *be a group. Then, the graph* $\mathcal{P}_e(G)$ *is connected if and only if* G *is periodic.*

Proof. Assume tha $\mathcal{P}_e(G)$ is connected and $x \in G\backslash\{1\}$. Then, x and 1 are connected. Hence, there exists a path $1 \sim x_1 \sim x_2 \sim \cdots \sim x_{n-1} \sim x_n = x$ in $\mathcal{P}_e(G)$. Since $1 \sim x_1$, there exists $k > 1$ such that $x_1^k = 1$. On the other hand, $x_1 \sim x_2$, it follows that $\langle x_1, x_2\rangle = \langle y\rangle$ is a cyclic subgroup of G. One has $x_1 \in \langle y\rangle$, so that $1 = x_1^k = y^{kl}$ for some $l \geq 1$. Hence, y has finite order. Since $x_2 \in \langle y\rangle$, x_2 has finite order. By induction one may prove that x has finite order. Consequently, G is periodic. The converse is clear. □

Corollary 2.4. *Let* $G_1, G_2, \ldots, G_n$ *be groups. Then,* $\mathcal{P}_e(G_1 \times G_2 \times \cdots \times G_n)$ *is connected if and only if* $\mathcal{P}_e(G_i)$ *is connected for every* i.

Corollary 2.5. *If* G *is an infinite cyclic group, then* $\mathcal{P}_e(G)$ *is not connected.*

Let G be a group. It is clear that the power graph $\mathcal{P}(G)$ is a spanning subgraph of the enhanced power graph $\mathcal{P}_e(G)$. However, the graph $\mathcal{P}(G)$ is not equal to the graph $\mathcal{P}_e(G)$ in general. The following question raises naturally.

Question 1. [1, Question 27] *For which (finite) groups is the power graph equal to the enhanced power graph?*

It was shown in [1, Theorem 28] that, for a finite G, the power graph $\mathcal{P}(G)$ is equal to the enhanced power graph $\mathcal{P}_e(G)$ if and only if every element of G has prime power order. As a corollary, if G is a finite p-group, then $\mathcal{P}(G) = \mathcal{P}_e(G)$. The following theorem shows that this result always holds for arbitrary p-groups.

Theorem 2.6. *If* G *is a* p*-group, then* $\mathcal{P}(G) = \mathcal{P}_e(G)$.

Proof. Assume that $x, y \in G$ such that $x \sim y$ in $\mathcal{P}_e(G)$. Then, $H = \langle x, y\rangle$ is a cyclic subgroup of G. It is clear that H is a finite cyclic p-subgroup of G. Hence, $\mathcal{P}(H) = \mathcal{P}_e(H)$. By [4, Theorem 2.4], $\mathcal{P}(H)$ is complete, which implies that $x \sim y$ in $\mathcal{P}(H)$. Therefore, $x \sim y$ in $\mathcal{P}(G)$, and so $\mathcal{P}(G) = \mathcal{P}_e(G)$. □

Recall that a field F is said to be *locally finite* if every its finitely generated subfield is finite. Equivalently, F is a locally finite field if and only if F is algebraic over its prime subfield $\mathbb{F}_p$ for some prime p.

Theorem 2.7. *Let* F *be a field. Then,* $\mathcal{P}(F^*) = \mathcal{P}_e(F^*)$ *if and only if* F *is a locally finite field.*

Proof. Assume that F is a locally finite field, and x is adjacent to y in $\mathcal{P}_e(F^*)$. We claim that x is adjacent to y in $\mathcal{P}(F^*)$. Let K be the subfield of F generated by x and y. By hypothesis, K is finite, and so K^* is cyclic. By [4, Theorem 2.4], $\mathcal{P}(K^*)$ is complete. Hence, $x \sim y$ in $\mathcal{P}(K^*)$, which implies that $x \sim y$ in $\mathcal{P}(F^*)$, so that $\mathcal{P}(F^*) = \mathcal{P}_e(F^*)$. Conversely, assume that $\mathcal{P}(F^*) = \mathcal{P}_e(F^*)$. Then, for every $x \in F^* \backslash \{1\}$, one has $x \sim 1$ in $\mathcal{P}(F^*) = \mathcal{P}_e(F^*)$. It follows that x has finite order, and so F^* is periodic. Hence, the characteristic of F must be a prime number p. Suppose that $\mathbb{F}_p$ is the prime subfield of F. It is clear that F is algebraic over $\mathbb{F}_p$. Consequently, F is a locally finite field. □

3. The connectivity of proper enhanced power graphs of p-groups

Let G be a p-group. Observe that G is a periodic group. In view of Theorem 2.2, the enhanced power graph $\mathcal{P}_e(G)$ is connected. However, the proper enhanced power graph $\mathcal{P}_e^*(G)$ is not connected in general. For example, for the 2-group $\mathbb{Z}_2 \times \mathbb{Z}_2$, one has $\mathcal{P}_e(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is connected; but $\mathcal{P}_e^*(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is not connected. In this section, we study the connectivity of the proper enhanced power graph of p-groups. We show that $\mathcal{P}_e^*(G)$ is connected if and only if G has a unique subgroup of order p. In particular, if G has finite order p^n, then $\mathcal{P}_e^*(G)$ is connected if and only if the group G is either cyclic or a 2-group isomorphic to the generalized quaternion group Q_{2^n}. We also prove that if $\mathcal{P}_e^*(G)$ is connected, then the graph $\mathcal{P}_e^*(\mathrm{Aut}(G))$ is connected provided p is odd.

Lemma 3.1. *Let G be a p-group and x a vertex of order p in $\mathcal{P}_e^*(G)$. Assume that C_x is a connceted component of $\mathcal{P}_e^*(G)$ containing x. Then, every $y \in C_x \backslash \{x\}$ is adjacent to x.*

Proof. By hypothesis, x is connected to y in $\mathcal{P}_e^*(G)$. Hence, there exists a path

$$x = x_0 \sim x_1 \sim x_2 \sim x_3 \sim \cdots \sim x_{n-1} \sim x_n = y$$

in $\mathcal{P}_e^*(G)$. By Theorem 2.6, it follows that $\mathcal{P}^*(G) = \mathcal{P}_e^*(G)$. Consider the path $x = x_0 \sim x_1 \sim x_2$. Then, either $\langle x \rangle \leq \langle x_1 \rangle$ or $\langle x_1 \rangle \leq \langle x \rangle$, and $\langle x_1, x_2 \rangle$ is cyclic. Since x has order p, $\langle x, x_2 \rangle \leq \langle x_1, x_2 \rangle$, and so $x \sim x_2$. Then, one has a path

$$x = x_0 \sim x_2 \sim x_3 \sim \cdots \sim x_{n-1} \sim x_n = y$$

in $\mathcal{P}_e^*(G)$. Repeating this argument with x_2 in place of x_1 and x_3 in place of x_2 shows that $x \sim x_3$. After a finite number of repetitions we have $x \sim y$. □

Theorem 3.2. *Let G be a nontrivial p-group. Then, there exists a one-to-one correspondence between connected components of $\mathcal{P}_e^*(G)$ and subgroups of order p. Hence, $\mathcal{P}_e^*(G)$ is connected if and only if G has a unique subgroup of order p.*

Proof. Let $\mathcal{S}$ be the set of all cyclic subgroups of order p of G and $\mathcal{T}$ be the set of all connected components of $\mathcal{P}_e^*(G)$. We would like to define $\varphi : \mathcal{S} \to \mathcal{T}$ by $\varphi(\langle x \rangle) = C_{\langle x \rangle}$, where $C_{\langle x \rangle}$ is the connected component of $\mathcal{P}_e^*(G)$ containing x. Suppose that $\langle x \rangle = \langle y \rangle$. Then, x is adjacent to y in $\mathcal{P}_e^*(G)$, which means that $C_{\langle x \rangle} = C_{\langle y \rangle}$. Therefore, the map $\varphi : \mathcal{S} \to \mathcal{T}$ given by $\varphi(\langle x \rangle) = C_{\langle x \rangle}$ is a well-defined function. To prove that φ is surjective, suppose C is a connected component of $\mathcal{P}_e^*(G)$. We choose a vetex z in C. Then, z has order p^k for some $k \geq 1$. It folllows that $x = z^{p^{k-1}}$ has order p. Observe that z is adjacent to x, which implies that $z \in C_{\langle x \rangle}$. It is easy to verify that $C = C_{\langle x \rangle} = \varphi(\langle x \rangle)$, and φ is surjective. To prove that φ is injective, suppose $\varphi(\langle x \rangle) = \varphi(\langle y \rangle)$. Then, $C_{\langle x \rangle} = C_{\langle y \rangle}$, so that x is

connected to y in $\mathcal{P}_e^*(G)$. By Lemma 3.1, $x \sim y$ in $\mathcal{P}_e^*(G) = \mathcal{P}^*(G)$. Since x and y has the same order p, $\langle x \rangle = \langle y \rangle$. Hence, φ is injective. The theorem is proved. □

Let p be a prime number. Recall that the *p-quasicyclic* group C_{p^∞} (known also as the *Prüfer group*) is the p-primary component of $\mathbb{Q}/\mathbb{Z}$. We define the group Q_{2^∞} as follows

$$Q_{2^\infty} = \bigcup_{i \geq 3} Q_{2^i},$$

where Q_{2^i} is a subgroup of index 2 in $Q_{2^{i+1}}$. We have the following result for infinite p-groups.

Theorem 3.3. *Let G be an infinite locally finite p-group. Then, $\mathcal{P}_e^*(G)$ is connected if and only if either $G \cong C_{p^\infty}$ or $G \cong Q_{2^\infty}$.*

Proof. Assume that $\mathcal{P}_e^*(G)$ is connected. Then, by Theorem 3.2, G has a unique subgroup of order p. According to Burnside's Theorem, G is a union of cyclic or generalized quaternion groups. If G is abelian, then $G \cong C_{p^\infty}$. If G is nonabelian, then $G \cong Q_{2^\infty}$. The converse is clear. □

Corollary 3.4. *If G is a finite noncyclic p-group, $\mathcal{P}_e^*(G)$ is not connected.*

Proof. Let n be the number of subgroups of order p in G. Then, $n \equiv 1 \ mod \ p$. Since G is noncyclic, $n > p$. According to Theorem 3.2, $\mathcal{P}_e^*(G)$ is not connected. □

Corollary 3.5. *Let G be a nontrivial finite p-group with $p > 2$. Then, $\mathcal{P}_e^*(G)$ is connected if and only if G is cyclic.*

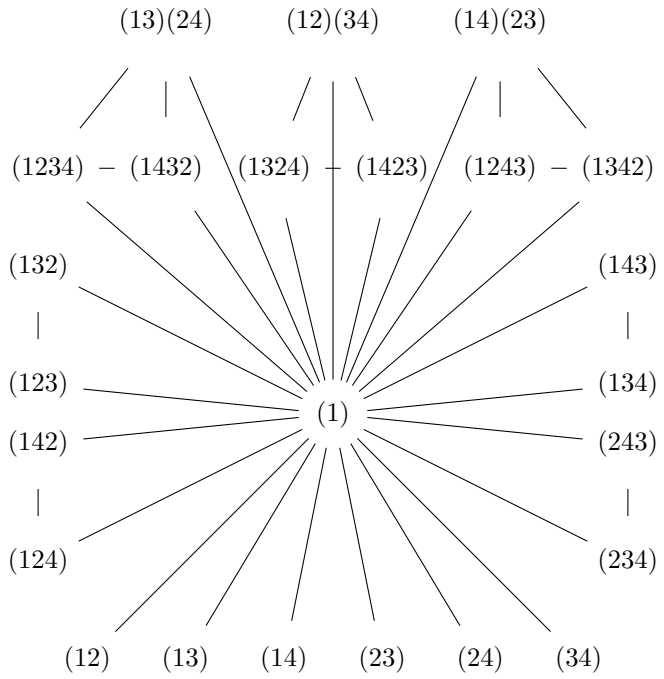
Proof. The result follows immediately from Theorem 3.2 and [25, Theorem 7.16]. □

Corollary 3.6. *Let G be a nontrivial finite 2-group. Then, $\mathcal{P}_e^*(G)$ is connected if and only if G is either cyclic or $G \cong Q_{2^n}$.*

Proof. The result follows immediately from Theorem 3.2 and [25, Theorem 7.17]. □

Lemma 3.7. *For the symmetric group S_4, the graph $\mathcal{P}_e(S_4)$ is connected, and $\mathcal{P}_e^*(S_4)$ is not connected.*

Proof. By Theorem 2.2, the graph $\mathcal{P}_e(S_4)$ is connected, and $\operatorname{diam}(\mathcal{P}_e(S_4)) = 2$. We can draw this graph as follows:



It follows that $\mathcal{P}_e^*(S_4)$ is not connected. □

Let G be a group. The *automorphism group* $\operatorname{Aut}(G)$ of G is the set of all automorphisms of G. It was known that $\operatorname{Aut}(G)$ is abelian provided G is cyclic. If G is a nontrivial finite cyclic p-group of order p^n, then $\operatorname{Aut}(G)$ has order $p^{n-1}(p-1)$.

Theorem 3.8. *Let G be a nontrivial finite p-group and $\mathcal{P}_e^*(G)$ be connected. Then,*

1. *If $p > 2$, then $\mathcal{P}_e^*(\operatorname{Aut}(G))$ is connected.*
2. *If $p = 2$ and $|G| = 2^n > 4$, then $\mathcal{P}_e^*(\operatorname{Aut}(G))$ is not connected.*

Proof. Assume that $\mathcal{P}_e^*(G)$ is connected. For $p > 2$, one has $\operatorname{Aut}(G)$ is cyclic. By Corollary 3.5, $\mathcal{P}_e^*(\operatorname{Aut}(G))$ is connected. Now, for $p = 2$ and $|G| = 2^n > 4$, according to Corollary 3.6, either G is cyclic or $G \cong Q_{2^n}$. If $G = \langle x \rangle$ is cyclic, then $\operatorname{Aut}(G)$ is an abelian group of order 2^{n-1}. One has $\operatorname{Aut}(G) = \langle \alpha \rangle \times \langle \beta \rangle$, where $\alpha(x) = x^{-1}$ and $\beta(x) = x^5$. Observe that α and $\alpha\beta^{2^{n-3}}$ are the only automorphisms of order 2. This means $\operatorname{Aut}(G)$ has more than one subgroup of order 2. Therefore, $\mathcal{P}_e^*(\operatorname{Aut}(G))$ is not connected. Now, suppose that $G \cong Q_{2^n}$. If $n = 3$, then

$$\operatorname{Aut}(G) \cong \operatorname{Aut}(Q_8) \cong S_4.$$

By Lemma 3.7, $\mathcal{P}_e^*(S_4)$ is not connected, and so $\mathcal{P}_e^*(\operatorname{Aut}(G))$ is not connected. Hence, we can assume that $n \geq 4$. Then, $\operatorname{Aut}(G) \cong \operatorname{Aut}(Q_{2^n})$. Observe that $\operatorname{Aut}(Q_{2^n})$ is a 2-group of order 2^{2n-3}, and $\operatorname{Aut}(Q_{2^n}) \cong H$, where

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{F}_{2^{n-1}} \right\}$$

is the subgroup of the general linear group $\mathrm{GL}_2(\mathbb{F}_{2^{n-1}})$. Observe that $x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ are two elements of order 2. This means $\langle x \rangle$ and $\langle y \rangle$ are two distinct subgroups of order 2 in H. Therefore, $\mathcal{P}_e^*(H)$ is not connected. This implies that $\mathcal{P}_e^*(\mathrm{Aut}(G))$ is not connected. The proof of the corollary is now complete. □

Remark 1. If G is a group of order 4, then either $G \cong C_4$ or $G \cong C_2 \times C_2$. If $G \cong C_4$, then the group $\mathrm{Aut}(G)$ has order 2, which implies that $\mathcal{P}_e^*(\mathrm{Aut}(G))$ has a unique vertex. If $G \cong C_2 \times C_2$, then $\mathrm{Aut}(G) \cong \mathrm{Aut}(C_2 \times C_2) \cong S_3$. Since $\mathcal{P}_e^*(S_3)$ is not connected, $\mathcal{P}_e^*(\mathrm{Aut}(G))$ is not connected.

4. The connectivity of the proper enhanced power graph of $\mathrm{GL}_n(D)$

In this section, we investigate the connectivity of the proper enhanced power graph $\mathcal{P}_e^*(G)$ of a group G. We show that if $\mathcal{P}_e^*(G)$ is connected, then G is either periodic or torsion-free. We also show the close relation between three the graphs; namely, the proper enhanced power graph $\mathcal{P}_e^*(G)$, the proper power graph $\mathcal{P}^*(G)$ and the cyclic subgroup intersection graph $\Gamma_c(G)$. Let D be an infinite division ring of characteristic not 2 and n a positive integer. We show that $\mathcal{P}_e^*(\mathrm{GL}_n(D))$ is connected if and only if D is a locally finite field.

For a periodic group G, the graph $\mathcal{P}_e(G)$ is always connected by Theorem 2.3. However, the graph $\mathcal{P}_e^*(G)$ is not necessarily connected. For example, for the symmetric group S_4, the graph $\mathcal{P}_e^*(S_4)$ is not connected by Lemma 3.7. Similarly, the power graph $\mathcal{P}(G)$ of the group G is connected; but the power power graph $\mathcal{P}^*(G)$ is not necessarily connected. The following question raises naturally.

Question 2. [1, Question 39] *Which groups do have the property that the proper power graph is connected?*

The next theorem shows that the graph $\mathcal{P}^*(G)$ is connected if and only if the graph $\mathcal{P}_e^*(G)$ is connected. Hence, Question 2 can be stated for the proper enhanced power graphs. For a nontrivial p-group G containing a unique subgroup of order p, the proper enhanced power graph $\mathcal{P}_e^*(G)$ is connected by Theorem 3.2.

Lemma 4.1. *Let G be a group and $x, y \in G$. Then, if $x \sim y$ in $\mathcal{P}_e^*(G)$, then either $x \sim y$ in $\mathcal{P}^*(G)$ or there exists $z \in G$ such that $x \sim z \sim y$ is a path in $\mathcal{P}^*(G)$.*

Proof. By hypothesis, the subgroup $\langle x, y \rangle$ is cyclic. It follows that there exists an element $z \in G$ such that $\langle x, y \rangle = \langle z \rangle$. If either $z = x$ or $z = y$, then either $\langle y \rangle \leq \langle x \rangle$ or $\langle x \rangle \leq \langle y \rangle$, and so $x \sim y$ in $\mathcal{P}^*(G)$. If z is different from x and y, then clearly $\langle x \rangle \leq \langle z \rangle$ and $\langle y \rangle \leq \langle z \rangle$. Hence, $x \sim z \sim y$ is a path in $\mathcal{P}^*(G)$. □

The following theorem shows that the connectivities of the proper enhanced power graph $\mathcal{P}_e^*(G)$ and the proper power graph $\mathcal{P}^*(G)$ of an arbitrary group G are always equivalent.

Theorem 4.2. *Let G be a group. Then, $\mathcal{P}_e^*(G)$ is connected if and only if $\mathcal{P}^*(G)$ is connected. Furthermore, if both $\mathcal{P}_e^*(G)$ and $\mathcal{P}^*(G)$ are connected, then*

$$\mathrm{diam}(\mathcal{P}_e^*(G)) \leq \mathrm{diam}(\mathcal{P}^*(G)) \leq 2\ \mathrm{diam}(\mathcal{P}_e^*(G)).$$

Proof. Assume that $\mathcal{P}^*(G)$ is connected. Since $\mathcal{P}^*(G)$ is a spanning subgraph of $\mathcal{P}_e^*(G)$, clearly $\mathcal{P}_e^*(G)$ is connected, and $\mathrm{diam}(\mathcal{P}_e^*(G)) \leq \mathrm{diam}(\mathcal{P}^*(G))$.

Conversely, assume that $\mathcal{P}_e^*(G)$ is connected. We claim that $\mathcal{P}^*(G)$ is connected, and $\operatorname{diam}(\mathcal{P}^*(G)) \leq 2 \operatorname{diam}(\mathcal{P}_e^*(G))$. To see this, suppose that x and y are two distinct vertices in $\mathcal{P}^*(G)$. Since $\mathcal{P}_e^*(G)$ is connected, there exists a path

$$x = x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_{n-1} \sim x_n = y$$

in $\mathcal{P}_e^*(G)$ with $d(x, y) = n$. According to Lemma 4.1, there exist distinct elements $z_1, z_2, \ldots, z_n \in G$ such that

$$x = x_0 \sim z_1 \sim x_1 \sim z_2 \sim x_2 \sim \cdots \sim x_{n-1} \sim z_n \sim x_n = y$$

is a path in $\mathcal{P}^*(G)$. This means $d(x, y) \leq 2n$ in $\mathcal{P}^*(G)$. Hence, $\mathcal{P}^*(G)$ is connected, and $\operatorname{diam}(\mathcal{P}^*(G)) \leq 2 \operatorname{diam}(\mathcal{P}_e^*(G))$. The proof of the theorem is now complete. □

Corollary 4.3. *Let G be a group. If the graph $\mathcal{P}_e^*(G)$ is connected, then G is either periodic or torsion-free.*

Proof. The proof follows immediately from Theorem 4.2 and [6, Lemma 1]. □

The following theorem shows that if G is a noncyclic group, then the connectivities of the proper enhanced power graph $\mathcal{P}_e^*(G)$ and the cyclic subgroup intersection graph $\Gamma_c(G)$ are equivalent.

Theorem 4.4. *Let G be a noncyclic group. Then, $\mathcal{P}_e^*(G)$ is connected if and only if $\Gamma_c(G)$ is connected. Furthermore, if both $\mathcal{P}_e^*(G)$ and $\Gamma_c(G)$ are connected, then*

$$\operatorname{diam}(\mathcal{P}_e^*(G)) \leq 2 \operatorname{diam}(\Gamma_c(G)) \leq 4 \operatorname{diam}(\mathcal{P}_e^*(G)).$$

Proof. Assume that $\mathcal{P}_e^*(G)$ is connected. We claim that $\Gamma_c(G)$ is connected. To see this, suppose that $\langle x \rangle$ and $\langle y \rangle$ are two distinct vertices in $\Gamma_c(G)$. It is clear that x and y are two distinct vertices in $\mathcal{P}_e^*(G)$. By hypothesis, there exists a path

$$x = x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_{n-1} \sim x_n = y$$

in $\mathcal{P}_e^*(G)$. For every $0 \leq i \leq n-1$, one has $\langle x_i, x_{i+1} \rangle = \langle z_i \rangle$ is cyclic subgroup of G. Without loss of generality, we can assume that $\langle x_i \rangle$ is not equal to $\langle x_{i+1} \rangle$. Then, one has $\langle x_i \rangle \sim \langle z_i \rangle \sim \langle x_{i+1} \rangle$ is a path in $\Gamma_c(G)$ for every $0 \leq i \leq n-1$. Then, we get a path

$$\langle x \rangle \sim \langle z_0 \rangle \sim \langle x_1 \rangle \sim \langle z_1 \rangle \sim \langle x_2 \rangle \sim \cdots \sim \langle x_{n-1} \rangle \sim \langle z_{n-1} \rangle \sim \langle y \rangle$$

in $\Gamma_c(G)$. Hence, $d(\langle x \rangle, \langle y \rangle) \leq 2d(x, y)$. Therefore, we can conclude that $\Gamma_c(G)$ is connected and $\operatorname{diam}(\Gamma_c(G)) \leq 2 \operatorname{diam}(\mathcal{P}_e^*(G))$.

Conversely, assume that $\Gamma_c(G)$ is connected. We shall prove that $\mathcal{P}_e^*(G)$ is connected. Suppose that a and b are two distinct vertices in $\mathcal{P}_e^*(G)$. If $\langle a \rangle = \langle b \rangle$, then clearly $\langle a, b \rangle$ is cyclic, and so $x \sim y$ in $\mathcal{P}_e^*(G)$. Hence, we may assume that $\langle a \rangle$ is not equal to $\langle b \rangle$. By hypothesis, there exists a path

$$\langle a \rangle = \langle a_0 \rangle \sim \langle a_1 \rangle \sim \langle a_2 \rangle \sim \cdots \sim \langle a_{n-1} \rangle \sim \langle a_n \rangle = \langle b \rangle$$

in $\Gamma_c(G)$. Observe that $a = a_0, a_1, a_2, \ldots, a_{n-1}, a_n = b$ are distinct vertices in $\mathcal{P}_e^*(G)$. For every $0 \leq i \leq n-1$, there exits a nonidentity element $c_i \in G$ such that $c_i \in \langle a_i \rangle \cap \langle a_{i+1} \rangle$. It follows that $\langle c_i \rangle \leq \langle a_i \rangle$ and $\langle c_i \rangle \leq \langle a_{i+1} \rangle$, and so $\langle c_i, a_i \rangle$ and $\langle c_i, a_{i+1} \rangle$ are cyclic. From this fact, we have $a_i \sim c_i \sim a_{i+1}$ is a path in $\mathcal{P}_e^*(G)$. Hence, we get a path

$$a = a_0 \sim c_0 \sim a_1 \sim c_1 \sim a_2 \sim \cdots \sim a_{n-1} \sim c_{n-1} \sim a_n = b$$

in $\mathcal{P}_e^*(G)$. Hence, $d(a, b) \leq 2d(\langle a \rangle, \langle b \rangle)$. Consequently, $\mathcal{P}_e^*(G)$ is connected and $\operatorname{diam}(\mathcal{P}_e^*(G)) \leq 2 \operatorname{diam}(\Gamma_c(G))$. The proof of the theorem is now complete. □

Combining Theorems 4.2 and 4.4, we get the following result.

Corollary 4.5. *Let G be a noncyclic group. Then, the following conditions are equivalent:*

(1) *The proper enhanced power graph $\mathcal{P}_e^*(G)$ is connected.*
(2) *The proper power graph $\mathcal{P}^*(G)$ is connected.*
(3) *The cyclic subgroup intersection graph $\Gamma_c(G)$ is connected.*

The following corollary gives the affirmative answer to Question 2 for the general skew linear groups.

Theorem 4.6. *Let D be an infinite division ring of characteristic not 2 and n a positive integer. Then, the following conditions are equivalent:*

(1) *The graph $\mathcal{P}_e^*(\mathrm{GL}_n(D))$ is connected.*
(2) *The graph $\mathcal{P}^*(\mathrm{GL}_n(D))$ is connected.*
(3) *The graph $\Gamma_c(\mathrm{GL}_n(D))$ is connected.*
(4) *D is a locally finite field.*

Proof. In the case $n = 1$. Then, by [19, Ex.13.4A], it follows that D^* is not cyclic. We claim that (1) is equivalent to (4). Indeed, assume that $\mathcal{P}_e^*(D^*)$ is connected. By Corollary 4.3, either D^* periodic or torsion-free. Since D has characteristic not 2, D^* must be periodic. By a result of Jacobson [12, Theorem 3.9.5], D is a field. It is easy to verify that D is a locally finite field. Conversely, suppose that D is a locally finite field. We claim that $\mathcal{P}_e^*(D^*)$ is connected. To see this, suppose x and y are two distinct vertices of $\mathcal{P}_e^*(D^*)$. Let K be the subfield of D generated by x and y. Then, K is a finite field, and so K^* is cyclic. Since x and y are in K^*, the subgroup $\langle x, y\rangle$ of K^* is cyclic, which implies that $x \sim y$ in $\mathcal{P}_e^*(D^*)$. Consequently, $\mathcal{P}_e^*(D^*)$ is connected. On the other hand, by Corollary 4.5, we get the conditions are equivalent. In the case when $n \geq 2$. Then, this result follows from Theorem 4.2 and [16, Theorem 5]. □

Remark 2. It is known that $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$. Since $\mathcal{P}_e^*(S_3)$ is not connected, $\mathcal{P}_e^*(\mathrm{GL}_2(\mathbb{F}_2))$ is not connected. By [16, Theorem 4], $\Gamma_c(\mathrm{GL}_2(\mathbb{F}_3))$ is not connected. Hence, $\mathcal{P}_e^*(\mathrm{GL}_2(\mathbb{F}_3))$ is not connected by Corollary 4.5. According to Corollary 4.5 and [16, Theorem 4], we get $\mathcal{P}_e^*(\mathrm{GL}_2(\mathbb{F}_9))$ is not connected.

References

[1] G. Aalipour, S. Akbari, P. J. Cameron, R. Nikandish and F. Shaveisi, On the structure of the power graph and enhanced power graph of a group, *Electron. J. Combin* **24**(3), paper 3.16 (2017).

[2] J. Abawajy, A. Kelarev, M. Chowdhury, Power graphs: a survey, *Electron. J. Graph Theory Appl,* **1**(2) (2013), 125-147.

[3] D. Alireza, E. Ahmad and J. Abbas, Some results on the power graphs of finite groups, *ScienceAsia* **41** (2015), 73-78.

[4] S. Bera and A. K. Bhuniya, On enhanced power graph, *J. Algebra Appl.* **17**(8) (2018), 8 pp.

[5] I. Bosnjak, R. Madarasz and S. Zahirovic, Some new results concerning power graphs and enhanced power graphs of groups. *arXiv preprint*, arXiv:2012.02851, (2020)

[6] P. J. Cameron and S. H. Jafari, On the connectivity and independence number of power graphs of groups, *Graphs and Combinatorics* **36** (2020), 895–904.

[7] P. J. Cameron, The power graph of a finite group II, *J. Group Theory* **13** (2010), 779–783.

[8] P. J. Cameron and S. Ghosh, The power graph of a finite group, *Disc. Math,* **311**(13) (2011), 1220–1222.

[9] P. J. Cameron, H. Guerra and S. Jurina, The power graph of a torsion-free group, *J. Alg. Combin,* **49** (2019), 83–98.

[10] I. Chakrabarty, S. Ghosh and M. K. Sen, Undirected power graphs of semigroups, *Semigroup Forum,* **78**(3) (2009), 410–426.
[11] T. T. Chelvam and M. Sattanathan, Power graph of finite abelian groups, *Alg. Disc. Math,* **16**(1) (2013), 33–41.
[12] P. M. Cohn, *Skew fields*, Cambridge University Press, 1995.
[13] A. Doostabadi, M. Farrokhi and D. Ghouchan, On the connectivity of proper power graphs of finite groups, *Commu. Alg,* **43** (2015), 4305–4319.
[14] E. Haghi and A. R. Ashrafi, Note on the cyclic subgroup intersection graph of a finite group, *Quasigroups and Related Systems*, **25** (2017), 245–250.
[15] A. Hamzed and A. R. Ashrafi, Automorphism groups of supergraphs of the power graph of a finite group, *European J. Combin*, **60** (2017), 82–88.
[16] B. X. Hai, B. X. B. Minh, L. V. Chua, M. H. Bien. *Low Diameter Algebraic Graphs.* Extended Abstracts EuroComb (2021). Trends in Mathematics, vol 14, (2021), 465-471. https://doi.org/10.1007/978-3-030-83823-2
[17] A. V. Kelarev and S. J. Quinn, A combinatorial property and power graphs of groups, *Contrib. General Algbera*, **12** (2000), 229–235.
[18] A. V. Kelarev, S. J. Quinn and R. Smolikova, Power graphs and semigroups of matrices, *Bull. Aust. Math. Soc*, **63** (2001), 341–344.
[19] T. Y. Lam, *Exercises in classical ring theory*, Second edition, Springer, 2003.
[20] X. Ma and Y. She, The metric dimension of the enhanced power graph of a finite group, *J. Alg. Appl*, **19**(1) (2020), 2050020.
[21] M. Mirzargar, A survey on the automorphism groups of the commuting graphs and power graphs, *Ser. Math. Inform*, **34**(4) (2019), 729–743.
[22] R. P. Panda and K. V. Krishna, On connectedness of power graphs of finite groups, *J. Alg. Appl*, **17**(10) (2018), 1850184 (20 pages).
[23] R. P. Panda, S. Dalal and J. Kumar, On the enhanced power graph of a finite group, *https://arxiv.org/abs/2001.08932.*
[24] R. Rajkumar and P. Devi, Intersection graphs of cyclic subgroups of groups, *Elec. Notes Discrete Math.* **53** (2016), 15–24.
[25] S. Roman, *Fundamentals of Group Theory*, An advanced approach, Birkhauser Boston, 2012.
[26] S. Zahirovic, I. Bosnjak and R. Madarasz, A study of enhanced power graphs of finite groups, *J. Alg. Appl.* (2019) 2050062 (20 pages).

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